ASYMPTOTIC BEHAVIOR OF CONTRACTIONS IN HILBERT SPACE

BY

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ABSTRACT

Let T be a contraction on a closed convex subset C of a Hilbert space H. It is proved that $T^n x/n$ tends to a limit y as n tends to infinity for every $x \in C$ and that the limit y is independent of x.

1. Introduction

Let H be a real Hilbert space and let D be a subset of H. A mapping $T: D \to D$ is called a contraction on D, or $T \in \mathrm{Cont}(D)$ if

$$|Tx - Ty| \leqq |x - y| \text{ for every } x, y \in D.$$

If $T \in \mathrm{Cont}(D)$ it has an obvious unique extension to $\bar{D}$ (the closure of D). Therefore there is no loss of generality in assuming, as we shall do henceforth, that D is closed. The object of this note is to study the behavior of $T^n x$ as $n \to \infty$ for $T \in \mathrm{Cont}(D)$. For clarity of exposition this note is restricted to contractions in Hilbert space. Many of the results however, hold also in uniformly convex Banach spaces. We start with two well known elementary lemmas.

LEMMA 1. *Let $C \subset H$ be closed and convex and let $x \in H$.*

a) *There exists a unique element $P_C x \in C$ with the property*

$$|P_C x - x| \leqq |y - x| \text{ for every } y \in C.$$

b) *$P_C x$ is characterized by the relations:*

$$P_C x \in C \text{ and } (x - P_C x, y - P_C x) \leqq 0 \text{ for every } y \in C.$$

c) *The mapping $x \to P_C x$ is a contraction on H.*

The proof of Lemma 1 can be found in almost every text on linear spaces. From Lemma 1 it follows in particular that every closed convex subset $C \subset H$

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has a unique element of minimum norm, namely $P_C 0$, we call it the minimal element of C.

LEMMA 2. *Let $C \subset H$ be closed and convex and let v be the minimal element of C. If $u_n \in C$ and $|u_n| \to |v|$ then $u_n \to v$ and $|u_n - v|^2 \leqq |u_n|^2 - |v|^2$.*

PROOF. Since v is the minimal element of C we have by Lemma 1(c) $|v|^2 \leqq (x,v)$ for every $x \in C$ and therefore

$$|u_n - v|^2 = |u_n|^2 - 2(u_n,v) + |v|^2 \leqq |u_n|^2 - |v|^2.$$

2. The main results

Let $D \subset H$ and let A be a mapping of D into H. We denote by conv(D) the convex hull of D and by $R(A)$ the range of A.

THEOREM 1. *Let $D \subset H$ be closed and let $T \in \mathrm{Cont}(D)$. If*

$$d_0 = \inf\{|y| : y \in \overline{\mathrm{conv}(R(I-T))}\} \text{ and } d_1 = \inf\{|y| : y \in \overline{R(I-T)}\} \text{ then}$$

$$d_0 \leqq \liminf_{n\to\infty} \frac{|T^n x|}{n} \leqq \limsup_{n\to\infty} \frac{|T^n x|}{n} \leqq d_1 \text{ for every } x \in D. \tag{1}$$

PROOF. Let $|x_m - Tx_m| \to d_1$. Since $T^n x_m = x_m + \sum_{k=1}^n (T-I)T^{k-1}x_m$ we have:

$$|T^n x_m| \leqq |x_m| + n|x_m - Tx_m|$$

or

$$\frac{|T^n x|}{n} \leqq \frac{|x_m| + |x - x_m|}{n} + |x_m - Tx_m|$$

and $\limsup_{n\to\infty} \frac{|T^n x|}{n} \leqq d_1$ follows easily.

Let v be the minimal element of $\overline{\mathrm{conv}(R(I-T))}$, $|v| = d_0$, and let $x \in D$, then

$$T^n x + nv = x + \sum_{k=1}^{n} (v - (I-T)T^{k-1}x);$$

multiplying this equality by v yields

$$(T^n x + nv, v) = (x,v) + \sum_{k=1}^{n} (v - (I-T)T^{k-1}x, v)$$

Since $(I-T)T^{k-1}x \in \overline{\mathrm{conv}(R(I-T))}$ we have $(v - (I-T)T^{k-1}x, v) \leqq 0, k = 1, \cdots n$ by the definition of v and therefore

$$(T^n x + nv, v) \leqq (x,v)$$

or after rearrangement

$$\frac{|T^n x|}{n} \geqq |v| - \frac{|x|}{n} = d_0 - \frac{x}{n}$$

and $\liminf_{n\to\infty} \frac{|T^n x|}{n} \geqq d_0$ follows.

COROLLARY 1. *If* $0 \in \overline{R(I-T)}$ *then for every* $x \in D$ $(T^n x/n) \to 0$ *as* $n \to \infty$.

DEFINITION. Let $K \subset H$ be closed and let v be the minimal element of $\overline{\mathrm{conv}(K)}$. The set K has the minimum property if $v \in K$.

THEOREM 2. *Let* $D \subset H$ *be closed and let* $T \in \mathrm{Cont}(D)$. *If* $\overline{R(I-T)}$ *has the minimum property then*

$$\lim_{n\to\infty} \frac{T^n x}{n} = -v \qquad \text{for every } x \in D \tag{2}$$

where v *is the unique element of minimum norm in* $\overline{R(I-T.)}$.

PROOF. Since

$$-\frac{T^n x}{n} + \frac{x}{n} = \frac{1}{n} \sum_{j=1}^{n} (I-T)T^{j-1}x \in \overline{\mathrm{conv}\,(R(I-T))} \tag{3}$$

and $\lim_{n\to\infty} \frac{|T^n x|}{n} = |v|$ by Theorem 1, we have $\lim_{n\to\infty} \frac{T^n x}{n} = -v$ by Lemma 2.

The main restriction in Theorem 2 is the fact that $\overline{R(I-T)}$ is required to have the minimum property. In the next section we shall exhibit some important examples in which this is indeed the situation. We conclude this section with a sufficient condition for $\overline{R(I-T)}$ to have the minimum property.

THEOREM 3. *Let* $D \subset H$ *be closed and let* $T \in \mathrm{Cont}(D)$. *If for some* $x \in D$ $T^n x - T^{n+1} x$ *is strongly convergent then* $\overline{R(I-T)}$ *has the minimum property.*

PROOF. Let $T^n x - T^{n+1} x \to v$. From (3) it then follows that $\frac{T^n x}{n} \to -v$. Now,

$$((I-T)T^n x - (I-T)u, T^n x - u) \geqq 0 \text{ for every } u \in D;$$

dividing by n and letting $n \to \infty$ we obtain

$$(v - w, v) \leqq 0 \qquad \text{for every } w \in R(I-T). \tag{4}$$

Clearly (4) is also true for every $w \in \overline{\mathrm{conv}(R(I-T))}$ and therefore v is the minimal element of $\overline{\mathrm{conv}(R(I-T))}$. But $v \in \overline{R(I-T)}$ since $(I-T)T^n x \to v$ and therefore $\overline{R(I-T)}$ has the minimum property.

3. Some applications

LEMMA 3. *Let $C \subset H$ be closed and convex and let $T \in \mathrm{Cont}(C)$, then for $\lambda > 0$ $R((I + \lambda)I - \lambda T)$ contains C. Moreover, if $T \in \mathrm{Cont}(H)$ then for every $\lambda > 0$ $R((I + \lambda)I - T) = H$.*

PROOF. To prove the first part we have to show that the equation $x = (1 + \lambda)y_\lambda - \lambda T y_\lambda$ has a solution $y_\lambda \in C$ for every $x \in C$. Define $\Psi_\lambda y = 1/(1 + \lambda)x + \lambda/(1+\lambda)Ty$. Clearly $\Psi_\lambda : C \to C$ and by Banach fixed point theorem Ψ_λ has a unique fixed point $y_\lambda \in C$ which is the desired solution. The second part is a simple consequence of the first part.

LEMMA 4. *Let $T \in \mathrm{Cont}(H)$ then $\overline{R(I - T)}$ is convex.*

PROOF. Let $x \in \mathrm{conv}(R(I - T))$ and let $\lambda > 0$, $x = (1 + \lambda)y_\lambda - Ty_\lambda$ (see Lemma 3) and $v = u - Tu$, then

$$((I - T)y_\lambda - (I - T)u,\ y_\lambda - u) \geqq 0$$

and therefore

$$(x - \lambda y_\lambda - v,\ y_\lambda - u) \geqq 0$$

which implies that $\lambda |y_\lambda|$ is bounded as $\lambda \to 0$. Let $\lambda_n y_\lambda \rightharpoonup \xi$ as $n \to \infty$, then

$$|\xi|^2 \leqq \limsup_{\lambda \to 0} |\lambda y_\lambda|^2 \leqq (x - v, \xi) \text{ for every } v \in R(I - T) \tag{5}$$

Clearly (5) holds also for every $v \in \overline{\mathrm{conv}(R(I - T))}$ and therefore in particular for $v = x$. Thus $\xi = 0$ and $\lambda y_\lambda \to 0$ so that $x \in \overline{R(I - T)}$ i.e. $\overline{R(I-T)} = \overline{\mathrm{conv}(R(I - T))}$ and $\overline{R(I - T)}$ is convex.

REMARK. Lemma 4 follows easily from the general theory of monotone operators. Clearly $I - T$ is monotone, continuous and defined on all of H. Therefore $I - T$ is maximal monotone (see e.g. Minty [5]) and the closure of the range of a maximal monotone operator is known to be convex (see e.g. Crandall and Pazy [3]).

Since a closed convex subset of H has the minimum property we have the following.

COROLLARY 2. *If $T \in \mathrm{Cont}(H)$ then for every $x \in H$, $T^n x/n \to -v$ where v is the minimal element of $\overline{R(I - T)}$.*

LEMMA 5. *Let $C \subset H$ be closed and convex and let $T \in \mathrm{Cont}(C)$ then $R(I-T)$ has the minimum property.*

PROOF. Let $T_1 = TP_C$ then $T_1 \in \mathrm{Cont}(H)$ and by Lemma 4 $\overline{R(I - T_1)}$ is convex. Let v be the minimal element of $\overline{R(I - T_1)}$, and let $v_n \in R(I - T_1)$, $v_n \to v$ and $v_n = x_n - T_1 x_n = x_n - TP_C x_n$. If $u_n = P_C x_n$ then

$$u_n \in R(I - T) \subset R(I - T_1) \subset \overline{R(I - T_1)}$$

and

$$(v_n - u_n, u_n) = (x_n - P_c x_n, P_C x_n - TP_C x_n) \geqq 0.$$

Therefore, $|u_n| \leqq |v_n|$ which implies $u_n \to v$ by Lemma 2. Thus $v \in \overline{R(I - T)}$. Since $\overline{\mathrm{conv}(R(I - T))} \subset \overline{R(I - T_1)}$ it is clear that v is the minimal element of $\overline{\mathrm{conv}(R(I - T))}$.

It would be interesting to know whether or not Lemma 5 is true in a uniformly convex Banach space.

COROLLARY 3. *Let $C \subset H$ be closed and convex and let $T \in \mathrm{Cont}(C)$ then for every $x \in C$, $\frac{T^n x}{n} \to -v$ where v is the unique element of minimum norm in $\overline{R(I - T)}$.*

REMARK. It is well known (see e.g. Minty [5]) that if $T \in \mathrm{Cont}(D)$, T has extensions $T_1 \supset T$ such that $T_1 \in \mathrm{Cont}(H)$. From Corollary 3 it follows that if C is closed and convex $T \in \mathrm{Cont}(C)$ and $T_1 \supset T$ is any extension of T to $\mathrm{Cont}(H)$ the element of minimum norm in $\overline{R(I - T_1)}$ is the same as the element of minimum norm in $\overline{R(I - T)}$.

The following result of Kirk [4] and Browder [1] follows easily from Corollary 3.

COROLLARY 4. *Let $C \subset H$ be closed, convex and bounded. If $T \in \mathrm{Cont}(C)$ then T has a fixed point in C.*

PROOF. Since $T \in \mathrm{Cont}\,(C)$ and C is bounded, $|T^n x|$ is bounded for every $x \in C$. By Corollary 3 we deduce $v = 0$ i.e. $0 \in \overline{R(I - T)}$. Let $x_n - Tx_n \to 0$. Since $x_n \in C$, $|x_n|$ is bounded. Let $x_{n_k} \rightharpoonup \xi$ then $(x_n - Tx_n - (y - Ty), x_n - y) \geqq 0$ implies $(y - Ty, \xi - y) \leqq 0$ for every $y \in C$. Choosing y to be the solution of $\xi = (1 + \lambda)y - \lambda Ty$ (see Lemma 3) we obtain $\lambda(y - Ty, y - Ty) \leqq 0$ i.e. $y = Ty$ and $y = \xi$. Thus T has a fixed point in C.

In [2] Browder and Petryshyn proved the following result.

THEOREM. *Let $T \in \mathrm{Cont}(H)$ then T has a fixed point if and only if there exists an element $x_0 \in H$ such that the sequence $\{|T^n x_0|\}$ is bounded.*

The following is a simple consequence of this result.

COROLLARY 5. *Let $T \in \mathrm{Cont}(C)$, $C \subset H$ closed and convex, then T has a* *fixed point in* C *if and only if there exists an element* $x_0 \in C$ *such that the sequence* $\{|T^n x_0|\}$ *is bounded.*

PROOF. It is clear that if T has a fixed point $x_0 \in C$ then $|T^n x_0| = |x_0|$ is bounded. On the other hand if for some $x_0 \in C$, $|T^n x_0|$ is bounded so is $|T^n x|$ for every $x \in C$ since $|T^n x| \leqq |x - x_0| + |T^n x_0|$. Let $T_1 = TP_C$ then $T_1{}^n x = T^n P_C x$ and therefore $|T_1{}^n x|$ is bounded and T_1 has a fixed point $x_0 \in H$ i.e. $x_0 = T_1 x_0$. But the range of T_1 is included in C and therefore $x_0 \in C$ and $x_0 = Tx_0$ i.e. T has a fixed point.

Let $C \subset H$ be closed and convex. For $T \in \mathrm{Cont}(C)$ we can summarize our results as follows.

COROLLARY 6. (a) $0 \in R(I - T)$ *if and only if* $|T^n x|$ *is a bounded sequence for every* $x \in C$.

(b) $0 \notin \overline{R(I-T)}$ *if and only if* $|T^n x|/n \to \alpha > 0$ *for every* $x \in C$.

(c) $0 \in \overline{R(I-T)}$ *but* $0 \notin R(I-T)$ *if and only if* $|T^n x| \to +\infty$ *and* $|T^n x|/n \to 0$ *for every* $x \in C$.

We conclude with simple examples of the three different possibilities. The situation (a) arises whenever C is bounded or else T has a fixed point in C. If T is a translation by a constant vector $x_0 \neq 0$ i.e. $Tx = x + x_0$ then we have the situation (b). Finally let $H = R^1$ and let

$$Tx = \begin{cases} x + 1 & \text{for } x \leqq 1 \\ x + \dfrac{1}{x} & \text{for } x \geqq 1 \end{cases}$$

It is not difficult to check that T is indeed a contraction on R^1 i.e. $T \in \mathrm{Cont}(H)$. Furthermore $R(I - T) = (-1, 0)$. Thus $0 \in \overline{R(I-T)}$ but $0 \notin R(I-T)$. It then follows from Corollary 6(c) that for every real $x_0 \geqq 1$ the sequence x_n defined by $x_{n+1} = x_n + 1/x_n$ is unbounded and satisfies $x_n/n \to 0$ as $n \to \infty$.

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